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EXAMPLES OF AMALGAMATED PRODUCTS

BY

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ABSTRACT

The rank of a group G is the minimal number of elements that generate G. For any natural number n we construct two groups, G_1 of rank $r(G_1) = n$ and G_2 of rank $r(G_2) = 2n$ such that their amalgamated product over an infinite cyclic subgroup, malnormal in both factors, is generated by $2n = r(G_1) + r(G_2) - n$ elements. We also consider an example of an amalgamated product of n factors: $G = \mathop{*}_{i=1}^{n}{}_A G_i$ such that $r(G) = n+1$, and $r(A) \geq 1$. This example realizes the lower bound given by Weidmann [W1] (see Theorem 2 in the present paper).

1. Introduction

By the celebrated Grushko's theorem, the rank of the free product $U_1 * U_2$ of two arbitrary groups U_1 and U_2 is $r(U_1 * U_2) = r(U_1) + r(U_2)$. However, no non-trivial analog of this theorem holds for amalgamated products. (See [L-S, I.11] and [Ro].) Kaufmann and Zieschang give in [K-Z] an example of an amalgamated product $U = U_1 *_C U_2$ such that $r(U) = r(U_1) = r(U_2) = n$, and the amalgamated subgroup C is cyclic of order two. They also prove the following.

THEOREM 1 (R. Kaufmann, H. Zieschang): *For every function $d\colon \mathbb{N} \to \mathbb{N}$ there exist groups U, V, W such that $r(U *_W V) \leq r(U) + r(V) - d(r(W))$.*

Weidmann [W2] shows that for any integer n, there are groups A, B such that $r(A) \geq n$, $r(B) \geq n$, and $r(A *_C B) = r(C) = 2$. [W2] and [K-Z] provide examples where the amalgamated subgroup is not malnormal. However, if we consider amalgamated products along a malnormal subgroup, the situation will change.

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Definition 1: Let H be a subgroup of a group G. H is **malnormal** in G if for any $g \in G \setminus H$, we have $gHg^{-1} \cap H = \{1\}$. We follow [W1] and say that $H \subset G$ is **premalnormal** in G if there exists a proper subgroup $U \subset G$ such that $H \subseteq U$ and for all $g \in G \setminus U$ the following condition holds: $gHg^{-1} \cap U = \{1\}$. Premalnormality is a weaker condition than malnormality: if H is malnormal in G, then putting $U = H$ we see that H is premalnormal also.

The rank of amalgamated products over (pre)malnormal cyclics is of special interest in knot theory because of the connection between the tunnel number $t(K)$ of a knot K and the rank of its fundamental group. In general, there is no uniform bound on the degeneration of tunnel numbers under connected sums, because using Morimoto's example [M], Kobayashi [Ko] shows that for any positive integer n there exist knots K_1 and K_2 such that the tunnel number $t(K_1 \# K_2)$ of their connected sum satisfies $t(K_1 \# K_2) < t(K_1) + t(K_2) - n$. However, some lower bounds are known. Scharlemann and Schultens [SS] prove that the tunnel number of the sum of n knots is at least n. Weidmann [W1] proves the following group theoretical analog of their result:

THEOREM 2 (R. Weidmann): *Let $G = \mathop{*}_{i=1}^{n} {}_A G_i$ and $A \neq \{1\}$. If A is premalnormal in G_i for $1 \leq i \leq n$, then $r(G) \geq n + 1$.*

This theorem implies the result of Scharlemann and Schultens; on the other hand, it follows that the inequality $r(G) \geq n + 1$ holds if A is malnormal also (which was proved in [KS] for the case $n = 2$). It is not difficult to see that in case A is cyclic, the estimate of Theorem 2 is precise: just take $G_i = \langle a, g_i \rangle$ to be a copy of the free group of rank 2 for $1 \leq i \leq n$, and $A = \langle a \rangle$. However, in the statement of Theorem 2, there is no restriction on the rank of the amalgamated subgroup A, while the above example (where $r(A) = 1$) cannot be extended to the general case. Nevertheless, as our Example 2 shows, the equality in the estimate of Theorem 2 can be attained in case $r(A) > 1$ as well.

Another lower bound for $r(G)$ is given by the following lemma.

LEMMA 1: *Let $G = \mathop{*}_{i=1}^{n} {}_A G_i$ be an amalgamated product of n groups, $A \neq \{1\}$, and suppose that $\mathcal{J} \subset \{1, \ldots, n\}$ is a non-empty set of indices. Let $H = \mathop{*}_{i \in \mathcal{J}} {}_A G_i$. If for all $i \notin \mathcal{J}$ there is a map $\varphi_i \colon G_i \to A$ such that $\varphi_i|_A = \mathrm{id}_A$, then $r(G) \geq r(H)$.*

Proof: We define a homomorphism $\varphi \colon G \to H$ in the following way:

$$\varphi|_{G_i} = \begin{cases} \mathrm{id}_{G_i}, & i \in \mathcal{J}, \\ \varphi_i, & i \notin \mathcal{J}. \end{cases}$$

Clearly, φ is onto. Thus $r(G) \geq r(H)$. ∎

It is well-known that every knot group is mapped onto the meridional subgroup by the abelianization, so Lemma 1 holds if we consider connected sum of n knots. In this case we get, in particular,

$$r(G) \geq \max_{1 \leq i \leq n} r(G_i).$$

Example 1 shows that equality can hold in the inequality of Lemma 1.

2. Examples of groups

LEMMA 2:

1. *Let a group be defined as follows:*

$$S = \langle g, h, z \mid z = [g, h] \rangle.$$

Then the subgroup $Z = \langle z \rangle \subset S$ is infinite cyclic and malnormal in S. Moreover, there exists a word $w(x, y)$ such that in the group

$$R = \langle z, b \mid b = w(z, bzb^{-1}) \rangle,$$

the subgroup $Z = \langle z \rangle \subset R$ is infinite cyclic and malnormal also.

2. *Let $W = \langle c, d \rangle \cong F_2$ be the free group of rank 2, and let $a_i = w_i(c, d)$, $i = 1, 2, \ldots, k$ be elements of W. Then the words w_i can be chosen in such a way that $A = \langle a_1, \ldots, a_k \rangle$ is a free subgroup of rank k which is malnormal in W.*

Proof: 1. By [Gr, page 136], a maximal cyclic subgroup Z of a torsion-free word hyperbolic group G is malnormal. The statement for the free group $S \cong F_2$ follows immediately because, in a free group, a commutator has no root. For the group R, the proof is constructive. We choose a word w so that R satisfies the small cancellation condition $C'(\frac{1}{6})$ (whence G is word hyperbolic), and so that there is a retract of R onto $Z = \langle z \rangle$ (which implies Z is a maximal cyclic subgroup of the torsion free group R). (See [Gr] and [L-S, Chapter V] for the definitions of a word hyperbolic group and of a group with small cancellation, respectively.) For instance, we set $w(x, y) = xy^{-1}x^2yx^3y^2xy^2x^2y^5x^6yxy^3$. Then any piece $p(x, y)$ is either of the following words (or their subwords): xyx, xy^2x, x^2yx, x^2y^2, xy^3, x^3y, yxy^2, y^2x^2, yx^3, and it is not difficult to see that R satisfies condition $C'(\frac{1}{7})$, and hence $C'(\frac{1}{6})$. Indeed, regard $w(x,y)$ and pieces $p(x,y)$ as functions of x and y. Substitute (as suggested by the statement)

$$(1) \qquad x = z, \quad y = bzb^{-1}.$$

Then the relation r of R is as follows:

$$\begin{aligned} r &= b^{-1}w(z, bzb^{-1}) \\ &= b^{-1}zbz^{-1}b^{-1}z^2bzb^{-1}z^3bz^2b^{-1}zbz^2b^{-1}z^2bz^5b^{-1}z^6bzb^{-1}zbz^3b^{-1}, \end{aligned}$$

so that $|r| = 46$. It can be checked that if p is a piece relatively to $\{r\}$, then $|p| \leq 6$. Hence

$$\frac{|p|}{|r|} \leq \frac{6}{46} < \frac{1}{7}$$

so that the given presentation of the group R satisfies $C'(\frac{1}{7})$ as claimed. The homomorphism φ: $R \to Z$ such that $\varphi(z) = z$ and $\varphi(b) = z^{29}$ is the retract.

2. Put $a_i = w_i(c,d) = cd^{r_i}cd^{r_i+1}\cdots cd^{s_i}$, $r_i = 25i$, $s_i = r_i + 24$. Because $[r_i, s_i] \cap [r_j, s_j] = \emptyset$ for $i < j$, a piece that can be cancelled in a product of a_i^{ϵ} and $a_j^{-\epsilon}$ ($i \neq j$, $\epsilon = \pm 1$) is either cd^{r_i} or cd^{s_i}. Hence $\{a_1, \ldots, a_k\}$ is a basis of a free subgroup of rank k.

Let $X_A = \{a_1^{\pm 1}, \ldots, a_k^{\pm 1}\}$ be an alphabet of A. We say that a word $f = x_1 \ldots x_l$ in the elements of X_A is **cyclically reduced in** A if f is reduced as a word in the alphabet $X_W = \{c^{\pm 1}, d^{\pm 1}\}$ of W and $x_1 \neq x_l^{-1}$. Notice that the word

$$a_1a_2^{-1} = cd^{r_1}cd^{r_1+1}\cdots cd^{s_1-s_2}c^{-1}\cdots d^{-(r_2+1)}c^{-1}d^{-r_2}c^{-1}$$

is cyclically reduced in A but is not cyclically reduced in the usual meaning.

Now, let $1 \neq u, v \in A$, and suppose that there exists $g \in W \setminus A$ such that $g^{-1}ug = v$. We claim that necessarily $g \in A$ which means malnormality of A. Let $v = x_1 \cdots x_n$, and let $u = y_1 \cdots y_m$ when $x_1, \ldots, x_n, y_1, \ldots, y_m \in X_A$. First, suppose that v is cyclically reduced in A. If u is not cyclically reduced in A, we write $u = s(u)\tilde{u}s(u)^{-1}$ where $s(u) \in A$, and $\tilde{u}$ (clearly, $\tilde{u} \in A$) is cyclically reduced in A. Now, $gug^{-1} = \tilde{g}\tilde{u}\tilde{g}^{-1}$ where $\tilde{g} = gs(u)$ so that $g \in A$ if and only if $\tilde{g} \in A$. Then $\tilde{g}\tilde{u}\tilde{g}^{-1}(= v)$ is a cyclic permutation of $\tilde{u}$. The exponents of d in the words w_i for all i are never repeated, and in any freely reduced product z of the elements of X_A the syllables $d^{r_i+1}, \ldots, d^{s_i-1}$ survive (as well as the very first and the very last exponents of d in z). Thus these syllables define both the factors and their order in z. The claim follows. If $v = s(v)\tilde{v}s(v)^{-1}$ (where $1 \neq s(v) \in A$ whence $\tilde{v} \in A$, and $\tilde{v}$ is cyclically reduced in A) is not cyclically reduced in A, then we consider the equality $\tilde{v} = \hat{g}\tilde{u}\hat{g}^{-1}$ where $\hat{g} = s(v)^{-1}\tilde{g}$ ($\tilde{g}$ is as above) so that $\hat{g} \in A$ if and only if $g \in A$. This completes the proof of the malnormality of A. ∎

2.1. Degeneration of the rank. For any $n \geq 3$ we construct two groups, G_1 of rank $r(G_1) = n$ and G_2 of rank $r(G_2) = 2n$ such that their amalgamated product $G = G_1 *_Z G_2$ (where $Z = \langle z \rangle \cong \mathbb{Z}$ is malnormal in both factors G_1 and G_2) is generated by $r(G_1) + r(G_2) - n = 2n = r(G_2)$ elements.

Example 1: Let

$$G_1 = \langle z, b_i z b_i^{-1} \mid b_i = w(z, b_i z b_i^{-1}),\ 1 \leq i \leq n-1 \rangle = \mathop{*}_{i=1}^{n} {}_Z R_i,$$

$$G_2 = \langle z, g_1, h_1, \ldots, g_n, h_n \mid z = [g_i, h_i] \text{ for all } i \rangle = \mathop{*}_{i=1}^{n} {}_Z S_i,$$

where $n \geq 3$, and R_i and S_i are copies of groups R and S respectively (R and S as in Lemma 2). Let $G = G_1 *_Z G_2$; then

$$X = \{g_n, h_n, b_1 g_1 b_1^{-1}, b_1 h_1 b_1^{-1}, \ldots, b_{n-1} g_{n-1} b_{n-1}^{-1}, b_{n-1} h_{n-1} b_{n-1}^{-1}\}$$

is a minimal system of generators of G.

Claim 1: *With the notation of Example 1, $r(G_1) = n$, $r(G_2) = 2n$, Z is infinite cyclic and malnormal in G_1 and G_2, and $r(G) = 2n = \#X$.*

Proof: The subgroup Z is infinite cyclic and malnormal in R and S, whence Z has the same properties in G_1 and G_2. The rank property of G_1 follows from Theorem 2. The group G_2 is generated by the $2n$ elements $g_1, h_1, \ldots, g_n, h_n$, and cannot be generated by a smaller number of elements, as the abelianization shows. All the generators of G_1 and G_2 (in the given presentations) can be expressed as words in the elements of X. Indeed, using the relations of G_2 we write $z = [g_n, h_n]$ and $b_i z b_i^{-1} = [b_i g_i b_i^{-1}, b_i h_i b_i^{-1}]$ for all $i = 1, \ldots, n-1$. Using the relations of G_1, we express $b_1, \ldots, b_{n-1}$ in terms of $z, b_i z b_i^{-1}$. Now all the generators of G_2 can be obtained from $b_i g_i b_i^{-1}, b_i h_i b_i^{-1} \in X$. Thus we have shown that $r(G) \leq \#X = 2n$. As $2n = r(G_2) = \max\{r(G_1), r(G_2)\}$ and the abelianization of G_1 is a retract of G_1 onto Z, in view of Lemma 1 we get $r(G) \geq 2n$, whence $r(G) = 2n (= r(G_1) + r(G_2) - n)$. ∎

2.2. The lower bound by R. Weidmann can be attained in general. Our next example shows that equality in the statement of Theorem 2 can hold when $r(A) > 1$ also. This example is very much in the spirit of examples constructed by Richard Weidmann for different purposes (see, for instance, [W3, Section 3]).

Example 2: We set $G_1 = W$ (W as in Lemma 2), and for all $j = 2, \dots, n$, $G_j = \langle a_1, \dots, a_k, g_j \rangle \cong F_{k+1}$, so that $A = \langle a_1, \dots, a_k \rangle \cong F_k$ is malnormal in G_i for all $i = 1, \dots, n$. Then G is generated by $c, d, g_2, \dots, g_n$, and $r(G) = n + 1$.

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